CONTACT PROBLEMS FOR BAR-REINFORCED STRIPS AND RECTANGULAR PLATES

PMM Vol. 39, № 3, 1975, pp. 559-564

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(Received August 27, 1973)

A solution of the mixed problem for an elastic strip reinforced by a semi-infinite flexible bar is obtained in closed form. The problem of the deformation of a rectangular plate, one of whose edges is reinforced by a bar, is reduced to a normal Poincaré-Koch system by the method of piecewise-homogeneous solution. Particular cases of this problem, a strip reinforced by a periodic system of bars, by one finite, or two semi-infinite bars, are examined.

The mixed problems for a half-plane welded to a single Melan bar of constant section have been examined in [1 – 3], where there are references to even earlier publications. Half-planes reinforced by a periodic system of bars have been investigated in [4, 5]. The first fundamental problem for a rectangle, one of whose sides is reinforced by an inflexible bar (the influence of the tangential contact stresses on the strain of the bar is also not taken account) is considered in [6], and a Melan bar in [7].

1. Let elastic Melan bar of constant section S, not inflexible, be welded to an elastic plate $-\infty < x < \infty$, $-1 \leqslant y \leqslant 1$ of thickness h at the sections $0 \leqslant x < -\infty$, $y = \pm 1$. Longitudinal forces T are applied to the bar endfaces $x = 0$, $y = \pm 1$ tangential forces $q(x)$ act on the free surface of the bar. It is required to determine the stresses in the plate which decay as $x \to -\infty$.

The boundary conditions for the plate in this problem are

$$\eta(x) \equiv h^{-1} S E_0 \partial^2 u / \partial x^2 + \tau_{xy} = -h^{-1} q(x) \quad (0 \leqslant x < \infty,\ y = -1) \tag{1.1}$$

$$\tau_{xy} = 0 \quad (-\infty < x < 0,\ y = -1), \qquad S E_0 \partial u / \partial x = T \quad (x = 0,\ y = -1) \tag{1.2}$$

$$\tau_{xy} = v = 0 \quad (-\infty < x < \infty,\ y = 0) \qquad \sigma_y = 0 \quad (-\infty < x < \infty,\ y = -1) \tag{1.3}$$

where E_0 is the elastic modulus of the bar. The solution is expressed by the formulas [8]

$$u(x, y) = \frac{1}{2\pi i} \int_L A(p) \chi(p) e^{px} dp, \qquad v(x, y) = \frac{1}{2\pi i} \int_L A(p) \zeta(p) e^{px} dp \tag{1.4}$$

$$\chi(p) = p[\varepsilon(p) - \rho(p)], \qquad \zeta(p) = \varepsilon'(p) + \rho'(p)$$

The contour L here lies in the strip $0 < \operatorname{Re} p < \delta$, the prime denotes the derivative with respect to y, and by virtue of (1.3) (ν and E are the elastic constants of the plate)

$$\rho(p) = 4E^{-1} \cos p \cos py, \quad \varepsilon(p) = 2E^{-1}(1 + \nu) p (y \cos p \sin py - \sin p \cos py)$$

It follows from conditions (1.1) and (1.2)

$$\tau^+(p) = -A(p) p^2 N_1(p), \qquad \eta^+(p) + \eta^-(p) = -A(p) p^2 N_2(p) \tag{1.5}$$

$$\tau^+(p) = \int_0^\infty \tau_{xy}(x, -1) e^{-px} dx, \qquad \eta^\pm(p) = \pm \int_0^{\pm\infty} \eta(x) e^{-px} dx$$

$$N_1(p) = \sin 2p + 2p, \quad N_2(p) = 2ap\cos^2 p + N_1(p), \quad a = 2E_0SE^{-1}h^{-1}$$

The zeros of the functions $N_1(p)$ and $N_2(p)$ are denoted, respectively, by a_k and b_k for $\operatorname{Re} p \geqslant 0$, $\operatorname{Im} p \geqslant 0$. Let us consider that $a_{-k} = -a_k$ and $b_{-k} = -b_k$. There are evidently no real and imaginary zeros except $a_0 = 0$ and $b_0 = 0$ among the a_k and b_k for large k

$$a_k = k\pi + iO(\ln k), \quad b_k = k\pi + iO(1) \tag{1.6}$$

Eliminating the function $A(p)$ in (1.5), we obtain the Wiener-Hopf equation [9]

$$\tau^+(p) = K(p)[\eta^+(p) + \eta^-(p)], \quad K(p) = N_1(p)N_2^{-1}(p)$$

Its solution is sought by the Gakhov formulas [10], as in [8], and it is $(p = \alpha + i\beta)$

$$\tau^+(p) = \tau_0^+(p)\left[-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\eta^+(t)\,dt}{\eta_0^-(t)(t-p)} + B\right], \quad -\pi < \arg(1 \pm p) < \pi \tag{1.7}$$

$$\tau_0^+(p) = a^{-1/2}(1+p)^{-1/2}\exp\left[\frac{p}{\pi}\int_0^\infty \frac{\ln[a(1+t^2)^{1/2}K(it)\,dt]}{t^2+p^2}\right]$$

$$\eta_0^-(p) = \frac{1}{\tau_0^+(-p)}$$

$$\tau^+(i\beta) = \frac{\tau_0^+(i\beta)}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{\eta^+(i\beta)}{\eta_0^-(i\beta)} - \frac{\eta^+(t)}{\eta_0^-(t)}\right]\frac{dt}{t-i\beta} + \frac{K(i\beta)}{2}\eta^+(i\beta) + B\tau_0^+(i\beta)$$

$$\tau_0^+(i\beta) = (1+i\beta)^{-1/2}(1+\beta^2)^{1/4}[K(i\beta)]^{1/2}\exp\left\{\frac{i\beta}{\pi}\int_0^\infty \ln\left[\frac{(1+t^2)^{1/2}K(it)}{(1+\beta^2)^{1/2}K(i\beta)}\right]\frac{dt}{t^2-\beta^2}\right\}$$

The quantity B is found from the bar equilibrium condition

$$\eta^+(0) + Th^{-1} = \tau^+(i\beta)|_{\beta=0}$$

It determines completely the intensity of the tangential stresses

$$\tau_{xy}(x, -1) = B(\pi a x)^{-1/2} + O(x^{1/2})$$

which grow without limit as $x \to +0$. In particular, if $q(x) = 0$, then

$$B = Th^{-1}(1 + {}^1/_2 a)^{1/2} \tag{1.8}$$

2. Now, let us remove a rectangular plate $-l_1 \leqslant x \leqslant l_2$, $-1 \leqslant y \leqslant 1$ $(l_1, l_2 > 0)$ from the strip. Let the previous conditions (1.1)–(1.3) remain on its longitudinal edges by retaining T and setting $q(x) = 0$ for simplicity. Let us assign normal stresses (displacements) and tangential displacements (stresses) which are symmetric in y to the plate endfaces, and let us apply identical longitudinal force T_2 to the right ends of the bar:

$$\tau_{xy}(-l_1, y) = f_1(y), \quad u(-l_1, y) = g_1(y)$$
$$\sigma_x(l_2, y) = f_2(y), \quad v(l_2, y) = g_2(y), \quad SE_0\frac{\partial u}{\partial x}\bigg|_{y=-1,\ x=l_2} = T_2 \tag{2.1}$$

Let us seek the solution in the form of series

$$u(x, y) = u^*(x, y) + \sum_{k=-\infty}^{\infty} u^k(x, y), \quad v(x, y) = v^*(x, y) + \sum_{k=-\infty}^{\infty}{}' v^k(x, y) \tag{2.2}$$

Here $u^*(x, y)$ and $v^*(x, y)$ are inhomogeneous displacements (1.4) in which the function $A(p)$ is expressed by (1.5), (1.7) and (1.8), $u^k(x, y)$ and $v^k(x, y)$ is a system of piecewise-homogeneous solutions of the problem (1.1) – (1.3) for $q(x) = 0,\ T = 0$

$$u^\circ(x, y) = A_0\left\{\frac{x}{E} - \frac{E_0 S(1 + {}^1/_2 a)^{1/2}}{2\pi i E h}\int\limits_L \frac{\tau_0^+(p)\,\chi(p)\,e^{px}\,dp}{p^2 N_1(p)}\right\} + B_0 \tag{2.3}$$

$$v^\circ(x, y) = -A_0\left\{\frac{\nu y}{E} + \frac{E_0 S(1 + {}^1/_2 a)^{1/2}}{2\pi i E h}\int\limits_L \frac{\tau_0^+(p)\,\zeta(p)\,e^{px}\,dp}{p^2 N_1(p)}\right\}$$

$$u^k(x, y) = \Phi^k\{\chi(p)\}, \qquad v^k(x, y) = \Phi^k\{\zeta(p)\}$$

$$\Phi^k\{f(p)\} = A_k \operatorname{Re} H_k[f(p)] + B_k \operatorname{Im} H_k[f(p)]$$

$$H_k[f(p)] = f(b_k)\,e^{b_k x} + b_k N_1(b_k)\,[\tau_0^+(b_k)]^{-1} I(b_k), \qquad k > 0$$

$$H_k[f(p)] = f(a_k)\,e^{a_k x} - a_k N_2(a_k)\,[\eta_0^-(a_k)]^{-1} I(a_k), \qquad k < 0$$

$$I(c) = \frac{1}{2\pi i}\int\limits_L \frac{\tau_0^+(p)\,f(p)\,e^{px}\,dp}{p(p - c)\,N_1(p)}$$

The elements of this system are constructed in the ordinary way [8]. In order to confirm whether they satisfy conditions (1.1) – (1.3), it is sufficient to replace the contour integrals by residue series in the zeros of the functions $N_1(p)$ and $N_2(p)$ according to the Cauchy theorem.

Let us determine the coefficients A_k and B_k. Using the Schiff orthogonality relationship [11], $(p_{k1} = a_k,\ p_{k2} = b_k)$

$$\int\limits_{-1}^{0} [\varepsilon'(p_{ks})\,\rho'(p_{ns}) + \varepsilon'(p_{ns})\,\rho'(p_{ks})]\,dy = 0 \qquad (p_{ks}^2 \neq p_{ns}^2)$$

we expand the given functions in series in the homogeneous solutions

$$g_1(y) = \sum_{k=1}^{\infty}[c_k\chi(a_k) + \bar{c}_k\chi(\bar{a}_k)] + c_0, \qquad g_2(y) = \sum_{k=1}^{\infty}[d_k\zeta(b_k) + \bar{d}_k\zeta(\bar{b}_k)] + d_0 y \tag{2.4}$$

$$c_k = a_k^{-1}N^{-1}(a_k)\int\limits_{-1}^{0}\left\{\left[f_1(y) - \frac{E}{1+\nu}\,g_1'(y)\right]\varepsilon'(a_k) + f_1(y)\,\rho'(a_k)\right\}dy$$

$$d_k = N^{-1}(b_k)\int\limits_{-1}^{0}\left\{f_2(y)\,[\rho(b_k) - \varepsilon(b_k)] - \frac{E}{1+\nu}\,g_2'(y)\,\varepsilon(b_k)\right\}dy - \frac{4SE_0\cos^2 b_k}{\nu EhN(b_k)}$$

$$d_0 = \nu^{-1}(Eh + SE_0)^{-1}\left[\int\limits_{-1}^{0} f_2(y)\,dy + SE_0T_2\right]$$

$$N(p) = \frac{2E}{1+\nu}\int\limits_{-1}^{0}\varepsilon'(p)\,\rho'(p)\,dy$$

The coefficients c_0 and B_0 do not influence the deformation of the plate.

Let us expand the functions $u^k(-l_1, y)$, $u^*(-l_1, y)$ and $v^k(l_2, y)$, $v^*(l_2, y)$ in residue series in the zeros of the functions $N_1(p)$ and $N_2(p)$, respectively, i.e. in series of the type

$$v^k(l_2, y) = \frac{1}{2}(A_k - iB_k)\left\{e^{b_k l_2}\zeta(b_k) + \sum_{n=-1}^{-\infty}\left[T_1(b_k, b_n)e^{b_n l_2}\zeta(b_n) + \right.\right.$$

$$\left.\left.T_1(b_k, \bar{b}_n)e^{\bar{b}_n l_2}\zeta(\bar{b}_n) + \frac{1}{2}(A_k + iB_k)\right]\right\}\left\{e^{\bar{b}_k l_2}\zeta(\bar{b}_k) + \right.$$

$$\left.\sum_{n=-1}^{-\infty}[T_1(\bar{b}_k, b_n)e^{b_n l_2}\zeta(b_n) + T_1(\bar{b}_k, \bar{b}_n)e^{\bar{b}_n l_2}\zeta(\bar{b}_n)]\right\}$$

$$T_1(s, p) = sN_1(s)\eta_0^-(p)[p(p-s)\tau_0^+(s)N_2^*(p)]^{-1}, \qquad k > 0,$$

where the asterisk denotes the derivative with respect to p. Let us substitute these expansions into (2.2) and then let us invert the order of summation with respect to k and n, in the double series in the left sides of conditions (2.1) for u and v. Let us substitute the series (2.4) instead of $g_1(y)$ and $g_2(y)$ in the right sides of these same conditions (2.1). Equating factors in y, we find

$$A_0 = -(\nu T + Ehd_0)(\nu Eh + E_0 S)^{-1} \tag{2.5}$$

Using the evenness relationship $\chi(-p) = -\chi(p)$, $\zeta(-p) = \zeta(p)$, equating factors in the functions $\chi(a_k)$, $\chi(\bar{a}_k)$, $\zeta(b_k)$, $\zeta(b_k)$ and introducing the unknowns

$$X_k - iY_k = (A_k - iB_k)\exp(l_2 b_k) \quad \text{for } k > 0 \tag{2.6}$$
$$X_k - iY_k = (A_k - iB_k)\exp(-l_1 a_k) \quad \text{for } k < 0$$

we obtain an infinite system of algebraic equations with a bilateral determinant

$$\varkappa X_k + \sum_{n=-\infty}^{\infty}{}' \{X_n \operatorname{Re}[\varphi_n(t_k) + \varphi_n(\bar{t}_k)] + Y_n \operatorname{Im}[\varphi_n(t_k) + \varphi_n(\bar{t}_k)]\} = -\operatorname{Re}\psi_k \tag{2.7}$$

$$\varkappa Y_k + \sum_{n=-\infty}^{\infty}{}' \{Y_n \operatorname{Re}[\varphi_n(t_k) - \varphi_n(\bar{t}_k)] - X_n \operatorname{Im}[\varphi_n(t_k) - \varphi_n(\bar{t}_k)]\} = \operatorname{Im}\psi_k$$

$$n \neq 0, \quad k = \pm 1, \pm 2, \ldots, \quad \varkappa = \operatorname{sign} k, \quad t_k = \begin{cases} b_k, & k > 0 \\ a_k, & k < 0 \end{cases}$$

$$\varphi_n(a_k) = T_2(b_n, -a_k)\exp(l_1 a_k - l_2 b_n) \quad (n > 0)$$

$$\varphi_n(b_k) = T_1(b_n, -b_k)\exp(-l_2 b_k - l_2 b_n) \quad (n > 0)$$

$$\varphi_n(a_k) = T_4(a_n, -a_k)\exp(l_1 a_k + l_1 a_n) \quad (n < 0)$$

$$\varphi_n(b_k) = T_3(a_n, -b_k)\exp(l_1 a_n - l_2 b_k) \quad (n < 0)$$

$$T_2(s, p) = -\frac{sN_1(s)\tau_0^+(p)}{p(p-s)\tau_0^+(s)N_1^*(p)}$$

$$T_3(s, p) = -\frac{sN_2(s)\eta_0^-(p)}{p(p-s)\eta_0^-(s)N_2^*(p)}$$

$$T_4(s, p) = \frac{sN_2(s)\tau_0^+(p)}{p(p-s)\eta_0^-(s)N_1^*(p)}, \quad \psi_k = -\frac{\eta_0^-(-b_k)(1 + 1/2a)^{1/2}}{b_k^2 e^{b_k l_2} N_2^*(b_k)} \times$$

$$\left(\frac{A_0 a}{2} + \frac{T}{h}\right) - d_k \quad (k > 0), \quad \psi_k = \frac{\tau_0^+(-a_k)e^{a_k l_1}(1 + 1/2a)^{1/2}}{a_k^2 N_1^*(a_k)} \times$$

$$\left(\frac{A_0 a}{2}+\frac{T}{h}\right)-c_{-k} \quad (k<0)$$

Because of the asymptotic estimates (1.6), the elements of the system (2.7) outside the diagonal decrease exponentially in the numbers of rows and columns. Therefore, the double series composed of them converges absolutely, and the system is normal according to Poincaré-Koch. Relying on the theory of normal systems [12], it can be shown that the normal solution X_k, Y_k exists, is unique, is determined by Cramer's rule and is estimated for large k by the asymptotic formulas ($\gamma < 2$)

$$X_k = -\operatorname{Re}\psi_k + O[k^\gamma \exp(-k\pi l_2)], \quad Y_k = \operatorname{Im}\psi_k + O[k^\gamma \exp(-k\pi l_2)] \quad (k>0) \tag{2.8}$$
$$X_k = \operatorname{Re}\psi_k + O[k^\gamma \exp(k\pi l_1)], \quad Y_k = -\operatorname{Im}\psi_k + O[k^\gamma \exp(k\pi l_1)] \quad (k<0)$$

It is expedient to keep m unknowns X_k, Y_k with negative and $E(ml_1l_2^{-1})$ positive numbers k in the system (2.7) in order to calculate the first unknowns X_k, Y_k ($E(x)$ denotes the integer part of x here). Then the rate of convergence of the solution X_k^m, Y_k^m of the truncated system to the exact solution is estimated by the formula

$$\max_k \{|X_k - X_k^m|, |Y_k - Y_k^m|\} = O\{|\psi_m| m^{2\gamma} e^{-m\pi l_1}\} \tag{2.9}$$

Elementary estimates based on (2.2), (2.3), (2.6) and (2.9) show that the greatest relative error δ in the solution (2.2) occurs at the endface of the rectangle $x=-l_1$ and is determined by the formula $\delta \sim \exp(-m\pi l_1)$. Therefore, for $\delta = 0{,}01$ and for $l_1 > 1/2$, say, it is sufficient to solve the system (2.7), truncated to $m=3$ and to use the asymptotic (2.8) for the remaining unknown X_k, Y_k.

If the functions $f_s(y)$ and $g_s(y)$ are twice continuously differentiable and satisfy consistency conditions with the magnitude of the longitudinal forces T_2 and the boundary conditions (1.1) – (1.3) at the corners of the plate, then the series (2.2) converges uniformly to the series (2.4) as $x \to -l_1, l_2$.

The method elucidated is broader and more effective than the method used in [8] in a simpler problem for a rectangle. The structure of the system (2.7) is improved considerably here; the possibility of expanding the arbitrary endface functions in the series (2.4) permits solution of the problem for rectangle with several kinds of boundary conditions by means of separation, i.e. with several bars on the lateral surface and with inhomogeneous conditions at the endfaces. The efficiency of the method has been verified also by computations on an electronic computer [13].

The nature of the singularities originating at the points of separation of the kinds of boundary conditions is evidently determined completely in all the problems of which we speak by the singularities in the individual elements of the system of piecewise-homogeneous solutions (2.3). Following the corresponding discussion from [14], it is easy to establish that as $x \to +0$

$$\tau_{xy}^{\circ}(x,1) = -\frac{A_0 E_0 S\sqrt{1+{}^1/_2 a}}{Eh\sqrt{\pi a x}} + O(\sqrt{x})$$
$$\tau_{xy}^{k}(x,1) = -(A_k \operatorname{Re} M_k + B_k \operatorname{Im} M_k)(\pi a x)^{-1/2} + O(\sqrt{x})$$
$$M_k = \frac{b_k N_1(b_k)}{\tau_0^+(b_k)} \quad (k>0), \qquad M_k = \frac{a_k N_2(a_k)}{\eta_0^-(a_k)} \quad (k<0)$$

Let us consider some particular cases of the endface conditions (2.1).

For $f_s(y) = g_s(y) = 0$ and $T_2 = 0$ the solution (2.2), (2.5), (2.7) determines the strain on a strip $-1 \leqslant y \leqslant 1$ reinforced by a periodic system of bars. Longitudinal forces $\pm T$, whose directions change from bar to bar, and act to one side are applied to the ends of each bar. Under these same conditions it is possible to put $l_1 = \infty$ or $l_2 = \infty$. In the first case $A_k = 0$ for $k < 0$. Of the four blocks in the system (2.7), one remains with the element numbers $k > 0$, $n > 0$. The solution determines the stresses in a strip on which a force $4T$ is transmitted through two bars of length $2l_2$ which are symmetric relative to the x-axis. The second case is the problem of a strip reinforced symmetrically by two semi-infinite bars. The spacing between the ends of the bars welded to one edge is $2l_1$, and the forces T applied are directed to opposite sides. For $k > 0$ in the solution (2.2) $A_k = 0$. The elements of three blocks vanish in the matrix of the system (2.7) because of the exponentials, and just a block with the element numbers $k < 0$, $n < 0$ remains.

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Translated by M. D. F.